Basic Definitions

Here are four basic terms for probability:

- **Event** – An event is a situation which could have different outcomes.
- **Outcome** – An outcome is a possible result of an event.
- **Sample Space** – All the possible outcomes. For example, the same space of rolling a die is $\{1, 2, 3, 4, 5, 6\}$ and the sample space of tossing a coin is $\{H, T\}$.
- **Random Event** – An event with equally likely outcomes.

Let's say a bag contains 4 red stones and 4 blue stones. One of the stones is pulled out

- The **event** is that a stone is selected from the bag.
- The **possible outcomes** are that a red stone or a blue stone could be selected. So the **sample space** is $\{R, B\}$.
- The event is a **random event** because there are the same number of red stones as blue stones.
  If there was only one blue stone, then the event would **not be random** because it is more likely that a red stone is drawn.

Relative Frequency

Each outcome has a **relative frequency** where

$$\text{Relative frequency} = \frac{\text{number of times outcome occurs}}{\text{total number of trials}}$$

So relative frequency can change from experiment to experiment. Here is an example:

a) A bag is full of 10 red, 10 blue and 10 green stones. Let's say 12 stones are drawn from the bag at random in this order:

$$\text{R G G B G G B B R R R G}$$

The relative frequency of red stones is $\frac{4}{12}$ or $\frac{1}{3}$.

The relative frequency of blue stones is $\frac{3}{12}$ or $\frac{1}{4}$.

The relative frequency of green stones is $\frac{5}{12}$.

b) The next time 14 stones are drawn, they are drawn in this order:

$$\text{B B G R R B R R B R B R G G}$$

The relative frequency of red stones is $\frac{6}{14}$ or $\frac{3}{7}$.

The relative frequency of blue stones is $\frac{5}{14}$.

The relative frequency of green stones is $\frac{3}{14}$.

Relative frequency is also sometimes called "experimental probability".

1. A coin is tossed once.

a What is the sample space (the possible outcomes)?

b Is this a random event? Why?

2. A single die is rolled ten times and these are the numbers it shows: $4, 5, 1, 4, 2, 6, 1, 3, 4, 1$

a What is the sample space of rolling a die and is rolling a die a random event? Why?

b In this experiment, what is the relevant frequency of a 2 being rolled?

c In this experiement, what is the relevant frequency of a 4 being rolled?

3. A box factory noticed that 6 out of 72 boxes were broken.

a What is the relative frequency of broken boxes?

b What is the relative frequency of unbroken boxes?

c If the box factory produced 3600 boxes, what number of these boxes is likely to be broken?

4. A fruit shop buys fruit according to this table:

Fruit	Number bought
Apples	81
Peaches	78
Apricots	84
Oranges	75
Bananas	82

a How many items of fruit were bought in total?

b What is the relative frequency of apricots?

c What is the relative frequency of oranges?

d The owner notices that 3 of the apples he bought were rotten. What is the frequency of rotten apples within the apples?

e If the owner bought 360 apples over a week, how many apples should they expect to be rotten?

Probability

A bag contains 1 red stone, 1 blue stone and 1 green stone. If you choose a stone without looking, what are the chances you will choose a red stone? The probability would be $\frac{1}{3}$.

This is the formula to find the probability of an outcome "X":

$$P(\mathrm{X}) = \frac{\text{number of ways X could occur}}{\text{total number of outcomes}}$$

The 'total number of outcomes' is also the size of the sample space. Here is an example:

A single die is rolled, answer the following questions

a. How many possible outcomes are there?

6 (the die could roll a 1, 2, 3, 4, 5 or 6)

b. What is the probability that a 4 is rolled?

$$P(4) = \frac{\text{number of ways to roll a 4}}{\text{total number of outcomes}}$$
$$= \frac{1}{6} \text{ or } 0.1\dot{6}$$

c. What is the probability that an even number is rolled?

$$P(\text{even}) = \frac{\text{number of even rolls}}{\text{total number of outcomes}} = \frac{3}{6}$$
$$= \frac{1}{2} \text{ or } 0.5$$

d. What is the probability that a number greater than 2 is rolled?

$$P(>2) = \frac{\text{number of rolls greater than 2}}{\text{total number of outcomes}} = \frac{4}{6}$$
$$= \frac{2}{3} \text{ or } 0.\dot{6}$$

The formula for probability could also be used in other ways.

A watchmaker selects 100 of his watches at random and notices 6 of them are faulty.

a. What is the relative frequency of faulty watches?

$$\frac{\text{number of faulty watches}}{100} = \frac{6}{100}$$

b. If the watchmaker made 1500 watches in total, how many would he expect are faulty?

$$P(\text{faulty}) = \frac{\text{number of faulty watches}}{\text{total number of watches}}$$

$$\frac{6}{100} = \frac{\text{number of faulty watches}}{1500}$$

$$\text{number of faulty watches} = 1500 \times \frac{6}{100} = 90$$

This will not **definitely** happen. It is just an expected value based on the probability.

Probability is Always Between 0 and 1

The number of ways any outcome X can occur will **always be less than or equal to** the total number of outcomes. So the **probability will always be a fraction**. This means that:

$$0 \leq P(\mathrm{X}) \leq 1$$

- If $P(\mathrm{X}) = 0$ it means that the outcome X is impossible. If $P(\mathrm{X}) = 1$, it means that the outcome X is certain.
- The closer $P(\mathrm{X})$ is to 0, the more unlikely X is and the closer $P(\mathrm{X})$ is to 1 the more likely X is.
- If you find a probability greater than 1 then a mistake has been made somewhere.

Complementary Probability

The complement of an outcome is when the outcome **doesn't** occur.

- The notation $\tilde{\mathrm{X}}$ ($\tilde{}$ is called *tilde*) means the **complement of** X and so $P(\tilde{\mathrm{X}})$ means the probability of X not occuring.
- The total probabilities of the outcomes must add up to 1 and so $P(\mathrm{X}) + P(\tilde{\mathrm{X}}) = 1$. This means we can use the formula:

$$P(\tilde{\mathrm{X}}) = 1 - P(\mathrm{X})$$

A bag contains 4 blue stones, 3 green stones and 5 red stones. Answer these questions if one stone is drawn at random

a. Find the probability that the stone will **not** be blue.

$$\begin{aligned} P(\text{not blue}) &= 1 - P(\text{blue}) \\ &= 1 - \frac{4}{12} \\ &= \frac{2}{3} \end{aligned}$$

b. Find the probability that the stone will **not** be red.

$$\begin{aligned} P(\text{not red}) &= 1 - P(\text{red}) \\ &= 1 - \frac{5}{12} \\ &= \frac{7}{12} \end{aligned}$$

c. Find the probability that the stone will **not** be green.

$$\begin{aligned} P(\text{not green}) &= 1 - P(\text{green}) \\ &= 1 - \frac{3}{12} \\ &= \frac{3}{4} \end{aligned}$$

1. A book has 120 pages and is opened to a random page.

a What is the probability it opens on page 89?

b What is the probability it is opened to an odd page?

c What is the probability of opening to page 65 or after?

d What is the probability of opening to a page after page 65?

e What is the probability that it is not opened to page 30?

f What is the probability that it is opened to a page number which is a multiple of 5?

g What is the probability that the page is **not** a multiple of 5?

h How are the answers from f and g related? Why?

2. You notice that when taking a test, you get 1 out of the first 12 questions incorrect.

a. What is the experimental probability of incorrect answers based on this information?

b. Based on this information, how many questions would you expect are incorrect if the test totalled 180 questions?

3. A standard deck of cards (with no jokers) is shuffled and placed face down and spread out.

a. If a card is drawn at random, what is the size of the sample space?

b. If a card is drawn at random, what is the probability it is an ace?

c. If a card is drawn at random, what is the probability it is the ace of spades?

d. If a card is drawn at random, what is the probability it is **not** a diamond?

e. If a card is drawn at random, what is the probability it is red?

Mutually Exclusive Events

Mutually **exclusive** events are events that cannot occur at the same time. If they are not mutually exclusive they are called **inclusive** events.

For example, rolling a 2 and rolling a 3 with a single die are mutually exclusive events – they can't happen at the same time.

If X and Y are mutually exclusive events then $P(\mathrm{X} \text{ or } \mathrm{Y}) = P(\mathrm{X}) + P(\mathrm{Y})$.

Find the probability of rolling a 2 or a 3 with a single die

$$\begin{aligned} P(2 \text{ or } 3) &= P(2) + P(3) \\ &= \frac{1}{6} + \frac{1}{6} \\ &= \frac{1}{3} \end{aligned}$$

Here is an example comparing mututally exclusive events and inclusive events:

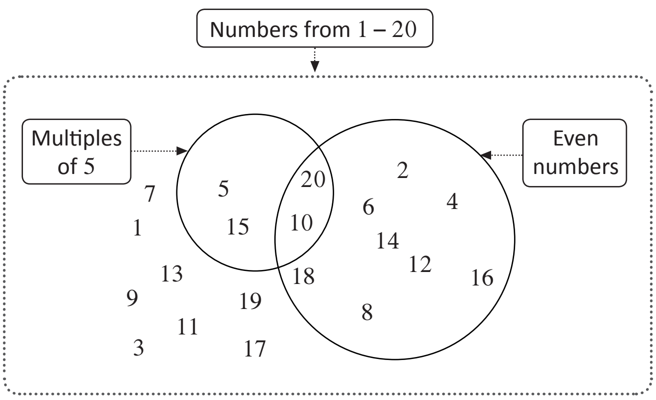
A book has 20 pages and is opened to a random page. Which of these are mutually exclusive?

X: The page number is a multiple of 5

Y: The page number is even

These are **inclusive events** because 10 and 20 are a multiples of 5 and even which means X and Y can happen at the same.

$$\therefore P(\mathrm{X} \text{ or } \mathrm{Y}) \neq P(\mathrm{X}) + P(\mathrm{Y})$$



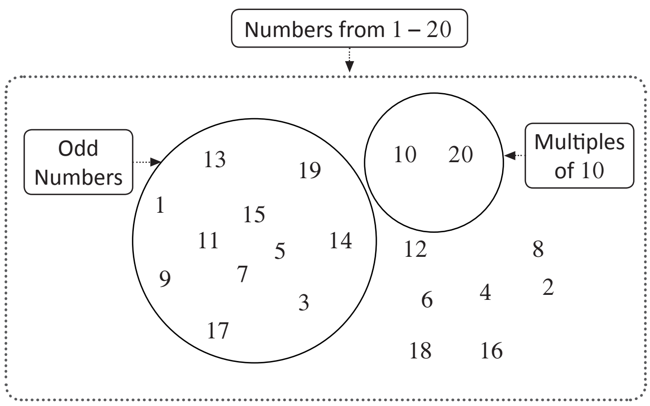
In the **Venn** diagram above it's easy to see that 10 and 20 are both multiples of 5 AND even numbers. The sets overlap.

X: The page number is a multiple of 10

Y: The page number is odd

These are mutually exclusive events because there are no numbers which are multiples of 10 and odd. So X and Y can't happen at the same time.

$$\begin{aligned} \therefore P(\mathrm{X} \text{ or } \mathrm{Y}) &= P(\mathrm{X}) + P(\mathrm{Y}) \\ &= \frac{2}{20} + \frac{10}{20} \\ &= \frac{3}{5} = 0.6 \end{aligned}$$



In the **Venn** diagram above it's easy to see that the sets of multiples of 10 and odd numbers are mutually exclusive. The sets do not overlap and are separate.

If X and Y are inclusive then there is an extra step to find $P(\text{X or Y})$. Subtract the probability of the 'overlapping' outcomes. To continue the example from the previous page.

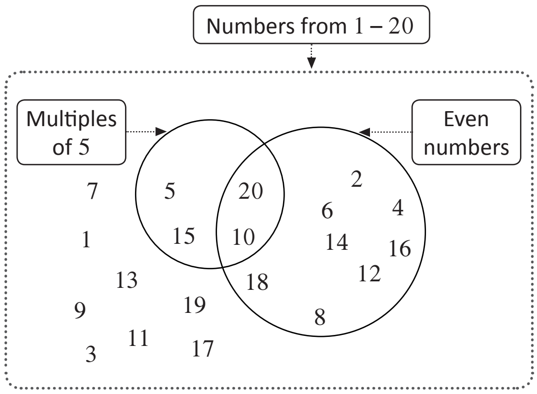
A book has 20 pages and is opened to a random page. Find the probability that the page number is even OR a multiple of 5

X: The page number is a multiple of 5

Y: The page number is even

$$\begin{aligned} P(\text{X or Y}) &= P(\text{X}) + P(\text{Y}) - P(\text{X and Y}) \\ &= \frac{4}{20} + \frac{10}{20} - \frac{2}{20} \\ &= \frac{12}{20} \\ &= \frac{3}{5} = 0.6 \end{aligned}$$

Since some page numbers are in both X and Y they shouldn't be counted twice



Compound Events

A compound event involves more than one outcome. It could have two stages or more. To find the probability of compound events, find the probability of each outcome and multiply them together.

A bag holds 6 red stones and 4 blue stones. Find the probability of drawing two blue stones from two draws.

Step 1: Find the probability that the first stone is blue.

$$\begin{aligned} P(\text{1st stone is blue}) &= \frac{\text{number of blue stones}}{\text{total stones}} \\ &= \frac{4}{10} \\ &= \frac{2}{5} \end{aligned}$$

Step 2: Find the probability that the second stone is blue.

$$\begin{aligned} P(\text{2nd stone is blue}) &= \frac{\text{number of blue stones remaining}}{\text{total stones remaining}} \\ &= \frac{3}{9} \\ &= \frac{1}{3} \end{aligned}$$

There is one less blue stone from the previous draw

There is one less stone in the bag from the previous

Step 3: Multiply the probabilities together:

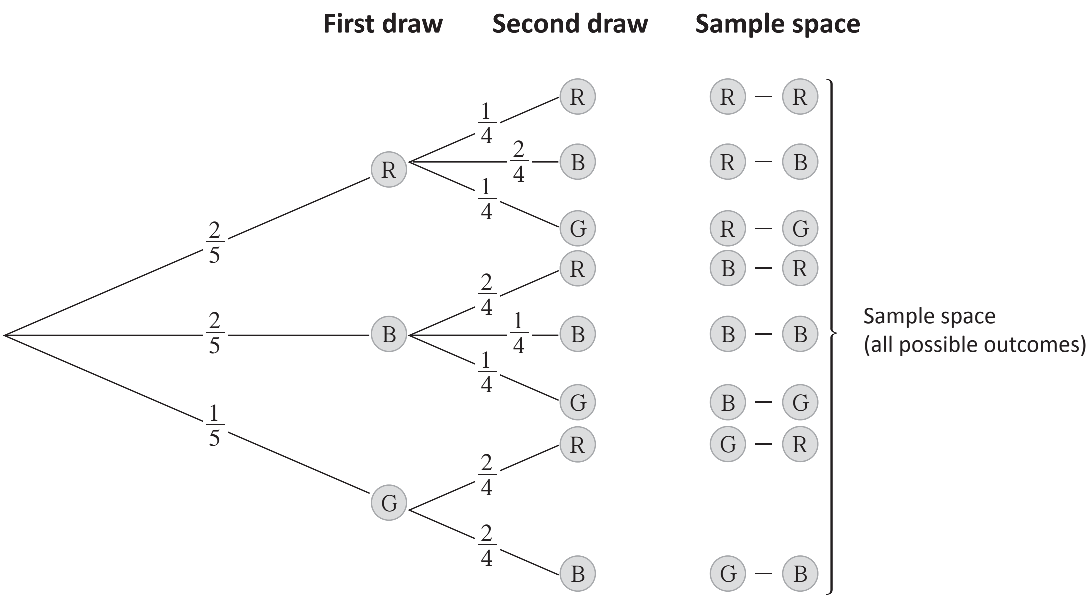
$$\begin{aligned} P(\text{2 blue stones}) &= P(\text{1st stone is blue}) \times P(\text{2nd stone is blue}) \\ &= \frac{2}{5} \times \frac{1}{3} \\ &= \frac{2}{15} \end{aligned}$$

Tree Diagrams

Tree diagrams are used with compound events to see all the possible outcomes (the sample space).

A bag contains 2 red, 1 green and 2 blue stone.

(a) What are all the possible methods to select two stones?



From the tree diagram we can that there are 8 possible outcomes in this sample space .

{RR, RB, RG, BR, BB, BG, GR, GB} (GG is not in the sample space because there is 1 green stone.)

(b) What is the probability that a blue stone is selected first and a red stone selected second? (without replacing the blue stone)

$$P(\text{BR}) = \frac{2}{5} \times \frac{2}{4} = \frac{1}{5}$$

(c) What is the probability of drawing one red and one green stone?

There are two possible outcomes with one red stone and one green table: GR and RG

$$\begin{aligned} P(\text{GR or RG}) &= P(\text{GR}) + P(\text{RG}) \\ &= \left(\frac{1}{5} \times \frac{2}{4}\right) + \left(\frac{2}{5} \times \frac{1}{4}\right) \\ &= \frac{4}{20} \\ &= \frac{1}{5} \end{aligned}$$

Notice the difference between (b) and (c). In (b), the order matters so there is only one way to draw blue first and red second. In (c), the order doesn't matter, so both GR and RG are counted.

Tables for Two-Stage Events

If the compound event is just a two-stage event, then a **two-way table** can be used.

Two multiple choice questions with options A, B and C need to be answered.

a. How many possible ways are there to answer the questions?

		$Q1$		
		A	B	C
$Q2$	A	AA	BA	CA
	B	AB	BB	CB
	C	AC	BC	CC

There are 9 possible ways to answer the $Q1$ and $Q2$: {AA, BA, CA, AB, BB, CB, AC, BC, CC}.
So the sample space size is 9.

b. What is the probability that both answers are A?

Only one outcome has both answers as A.

$$\begin{aligned}\therefore P(\text{both answers are A}) &= P(\mathrm{AA})\\ &= \frac{1}{9}\end{aligned}$$

c. What is the probability that both answers are the same?

$$\begin{aligned}P(\text{same answers}) &= P(\mathrm{AA}\text{ or }\mathrm{BB}\text{ or }\mathrm{CC})\\ &= P(\mathrm{AA}) + P(\mathrm{BB}) + P(\mathrm{CC})\\ &= \frac{1}{9} + \frac{1}{9} + \frac{1}{9}\\ &= \frac{1}{3}\end{aligned}$$

d. What are the chanced that the answers are different?

$$\begin{aligned}P(\text{different answers}) &= 1 - P(\text{same answers})\\ &= 1 - \frac{1}{3}\\ &= \frac{2}{3}\end{aligned}$$

To find the sample size – of a compound event – without a table or tree diagram, multiply the sample sizes of each stage. In the example above there are 3 ways to answer $Q1$ and 3 ways to answer $Q2$, so the sample size is $3 \times 3 = 9$.

A restaurant serves 5 starters, 4 mains and 3 desserts. How many ways are there to order a three course meal of a starter, main, and dessert?

$$\begin{aligned}\text{starters} \times \text{mains} \times \text{desserts} &= 5 \times 4 \times 3\\ &= 60 \text{ different ways to order a three course meal.}\end{aligned}$$

1. What is the difference between mutually exclusive events and inclusive events?

2. Identify in the following if outcomes A and B are mutually exclusive or not. Give a reason why you say so.

a. A: Obtaining 'heads' from a coin toss

B: Obtaining 'tails' from a coin toss

b. A: Finishing a task between Monday and Thursday

B: Finishing a task between Saturday and Tuesday

3. A single die is rolled. Answer the questions about these outcomes (check if they are mutually exclusive first):

A: Rolling a 1 or a 6

B: Rolling an even number

C: Rolling an odd number

D: Rolling a 3

a. Find P (B or D).

b. Find P (A or D).

c. Find P (A or C).

d. Find P (B or C).

4. Use this information to answer the following questions:

- $P(\text{A}) = \frac{1}{2}$
- $P(\text{B}) = \frac{3}{10}$
- $P(\text{C}) = \frac{1}{5}$
- $P(\text{D}) = \frac{2}{25}$
- $P(\text{A or B}) = \frac{4}{5}$
- $P(\text{C or D}) = \frac{27}{100}$
- $P(\text{B or C}) = \frac{11}{25}$

a Are A and B mutually exclusive?

b Are C and D mutually exclusive?

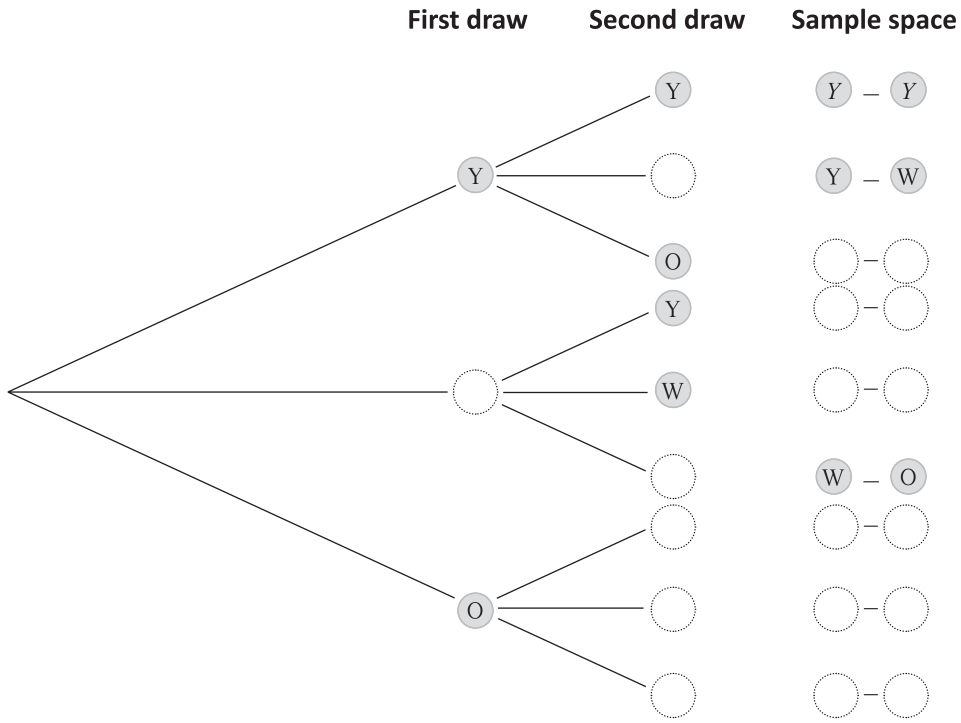
c Are B and C mutually exclusive?

d If $P(\text{A or C}) = \frac{3}{5}$,
use $P(\text{A or C}) = P(\text{A}) + P(\text{C}) - P(\text{A and C})$
to find $P(\text{A and C})$.

e Find $P(\text{B and C})$.

7. A bag contains 1 yellow, 1 white and 1 orange stone. A stone is drawn at random and then replaced. Then a stone is drawn at random for a second time.

a Complete the tree diagram below for this compound event:



b How big is the sample space of this experiment? Is this what you expected?

c What is the probability the white stone will be drawn first?

d What is the probability the white stone will be drawn second?

e What is the probability that the white stone will be drawn both times?

f Redraw the tree diagram if the stone that is drawn first is not replaced?

g What is the size of the sample space now?

h What is the probability that the yellow stone will be drawn first?

i What is the probability that the yellow stone will be drawn second?

j What is the probability that the yellow stone is drawn both times?

8. A tennis tournament has a singles trophy and a doubles trophy. The countries competing for the singles trophy are: India, Spain and Greece. The countries competing for the doubles trophy are just India and Spain. Each country has equal chance to win the trophies.

a Draw a table for this two-stage event of trophy winners.

b What is the probability that Greece will win the singles trophy?

c What is the probability that India will win both trophies?

d What is the probability that India and Spain will win a trophy each?

e What is the probability that Spain and Greece will win a trophy each?

f What is the probability that India will not win a trophy?